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THE MINIMUM NUMBER OF LINEAR DECISION FUNCTIONS TO IMPLEMENT A DECISION PROCESS

TECHNICAL DOCUMENTARY REPORT NO. ESD-TDR-64-171

DECEMBER 1964

H. Joksch

D. Liss

Prepared for

DIRECTORATE OF COMPUTERS
ELECTRONIC SYSTEMS DIVISION
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
L. G. Hanscom Field, Bedford, Massachusetts

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THE MITRE CORPORATION
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ABSTRACT

This document determines a lower limit on the number of linear decision functions necessary to place a certain number of objects into a certain number of categories. This lower limit is a function of the number of parameters necessary to describe each object and the number of categories.

REVIEW AND APPROVAL

This technical documentary report has been reviewed and is approved.

James W. Woodward for Capt

for SEYMOUR JEFFERY
Major, USAF
Chief, Computer Division
Directorate of Computers
Deputy for Engineering & Technology

SECTION I

INTRODUCTION

A decision problem can be considered as a problem in classifying objects according to their characteristics. The following are simple examples of decisions to be made:

> Into which category shall a book be placed according to its contents?
>
> Into which category shall the outcome of a statistical test be placed depending upon the numerical result (shall the hypothesis be accepted or rejected)?
>
> Which action shall be taken in a military situation described by certain characteristics?

This list could arbitrarily be extended to demonstrate that all decision problems are basically problems of classification. For decisions under risk or uncertainty, some finer points have to be introduced, but the basic structure of the problem remains unchanged.

SECTION II

DISCUSSION

The characteristics of the objects to be classified can be either yes/no (binary valued) statements of possessing or not possessing a certain property, they can be indications of which of a discrete set (integral valued) of possibilities applies, or they can be one of a continuous set (real valued) which describes properties of the object quantitatively. For the different properties of one object, any one of the three named cases may be applicable. In this report it is assumed that each property of the object is given by a real number.

The least restrictive way of classification is to have a complete listing of all objects with an indication of the class to which each belongs. In most cases, such a listing would be extremely long, if not infinitely long, and therefore impractical or impossible to use. Classifications are rarely given in this manner. Usually, objects are classified according to functions of their parameters. Therefore, practical decisions generally consist of evaluating certain functions of the characteristics; namely a mapping of the characteristics space into a decision space.

If the classification is given by a complete listing of objects, any completely arbitrary classification is possible. If the classification is done by evaluating "reasonable" functions, then it is generally the case that classes correspond in a 1-1 manner to regions in the decision space. In many cases, it holds that with any two objects which belong to one class, all objects "between" them belong to the same class. This is the same as the mathematical statement that the classes can be described by non-intersecting convex sets in the characteristics space. One of the important properties of convex sets is

that any two of them, which do not intersect, can be separated by a hyperplane. Therefore, we can separate all classes of these objects by a sufficient number of hyperplanes. Separation by a hyperplane is mathematically equivalent to the evaluation of a linear function, which is the type of function most easily handled mathematically. Therefore, under reasonable restrictions, a decision problem can be reduced from an extensive search procedure, through a complete listing or evaluation of arbitrary functions, to the evaluation of linear functions which are easily handled.

The best means of determining these linear functions is a problem which will be dealt with in a future report. This present report is concerned with the problem: How <u>many</u> linear functions are needed to represent a certain number of classes? This number is dependent upon the specific configuration of the convex sets. However, it is worthwhile to know at least upper and lower bounds for this number.

UPPER LIMIT

The upper bound is easily obtained. If, between any two out of the R regions, a separating hyperplane is used, then all classes are separated. Therefore this number is $(R/2)$. The lower bound is more difficult to obtain.

LOWER LIMIT

If the maximum number of regions $R(m, n)$ into which an n-dimensional space can be divided by m hyperplanes can be determined, then this would give a lower limit to the number m of hyperplanes required to separate $R(m, n)$ regions in an n-dimensional space.

This maximum number of regions

$$R(m,\ n) = \sum_{k=0}^{n} \binom{m}{k} \qquad m \geq 1,\ n \geq 1\ . \tag{1}$$

The following proof uses essentially the argument of Winder,[1] who determined the maximum number of regions $N(m,\ n)$ obtained by m hyperplanes in an n-dimensional Euclidean space when all hyperplanes intersect in one point to be

$$N(m,\ n) = 2 \sum_{k=0}^{n-1} \binom{m-1}{k} \qquad n \geq 1,\ m \geq 1\ . \tag{2}$$

The same result has been obtained by Cameron.[2] *

In order to obtain the maximum number of regions, the hyperplanes cannot have completely arbitrary positions. It is defined that m hyperplanes in an n-dimensional Euclidean space are in <u>independent position</u>, if every subset of $k \leq n$ of them has an $n-k$ dimensional linear manifold as an intersection, and any intersection of $n+1$ is empty. For any m and n, hyperplanes exist in independent position, since this is equivalent to the statement that there are sets of m linear equations in n variables such that all subsets of $k \leq n$ equations have exactly $n-k$ dimensional solution spaces and all subsets of $n+1$ have no solution.

*Cameron proves only that the maximum number cannot be larger than this expression, because he counts the combinatorial possiblities without regard to the possibilities of their geometric realization. Winder seems to be aware of this point as his definition of "general position" shows, but does not mention the problem explicitly.

Theorem:

The maximum number of regions into which an n-dimensional Euclidean space can be divided by m hyperplanes is

$$R(m, n) = \sum_{k=0}^{n} \binom{m}{k} \tag{3}$$

and this number is obtained by m hyperplanes in independent position.

Proof:

Since m points divide a line into $m+1$ parts, $R(m, 1) = m+1$ which equals

$$\sum_{k=0}^{1} \binom{m}{k} = \binom{m}{0} + \binom{m}{1} = m+1 \ . \tag{4}$$

One hyperplane divides any Euclidean space into two parts, $R(1, n) = 2$, which equals

$$\sum_{k=0}^{n} \binom{1}{k} = \binom{1}{0} + \binom{1}{1} = 2. \tag{5}$$

Starting with these values, the theorem can be proved by induction. Assume that $m-1$ hyperplanes in independent positions are given; they divide the space into the maximal number of $R(m-1, n)$ regions. Any m^{th} hyperplane intersects at most all the other $m-1$ hyperplanes, and if all m are in independent position, then the m^{th} one intersects all the other $m-1$ in

(n-2)-dimensional subspaces. These subspaces are hyperplanes in independent position with the (n-1)-dimensional space given by the m^{th} hyperplane. Each of the R(m-1, n-1) sections, into which this hyperplane is divided, divides one of the regions (determined by the m-1 hyperplanes) in the n-dimensional space into two regions, adding R(m-1, n-1) to the previous R(m-1, n) regions. The result is the recurring formula

$$R(m, n) = R(m-1, n) + R(m-1, n-1). \tag{6}$$

This formula has exactly the same structure as that obtained by Winder and Cameron. From the inductive hypothesis

$$R(m-1, n) = \sum_{k=0}^{n} \binom{m-1}{k}, \tag{7}$$

$$R(m-1, n-1) = \sum_{k=0}^{n-1} \binom{m-1}{k} = \sum_{k=0}^{n} \binom{m-1}{k-1}, \tag{8}$$

and the identity

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}, \tag{9}$$

the following is obtained

$$R(m-1, n) + R(m-1, n-1) = \sum_{k=0}^{n} \left[\binom{m-1}{k} + \binom{m-1}{k-1}\right] = \sum_{k=0}^{n} \binom{m}{k} = R(m, n) . \tag{10}$$

This proves out the theorem.

SOME NUMERICAL VALUES

Table I contains the values of $R(m, n)$ for small values of m and n. Table II contains the values of $N(m, n)$ as given by Cameron, for some small values of m and n.

Table I: $R(m, n)$

m \ n	1	2	3	4	5
1	2	2	2	2	2
2	3	4	4	4	4
3	4	7	8	8	8
4	5	11	15	16	16
5	6	16	26	31	32
6	7	22	42	57	63

Table II: $N(m, n)$

m \ n	1	2	3	4	5
1	2	2	2	2	2
2	2	4	4	4	4
3	2	6	8	8	8
4	2	8	14	16	16
5	2	10	22	30	32
6	2	12	32	52	62

The values above the main diagonals have to be the same in both cases since, for no more than n hyperplanes in an n-dimensional space, it is not necessary that all hyperplanes go through one point.

The values below the main diagonal fit the following pattern:

$$N(m, n) = 2R(m-1, n-1) \tag{11}$$

which is explained by a comparison of the formulas (1) and (2). Furthermore,

$$R(m, n) = N(m, n) + \binom{m-1}{n} \tag{12}$$

since the parts of the schemes below the diagonal differ by a column and diagonal of one, and that the recursion formula (6) is that of Pascal's triangle. However, there seems to be no simple geometric interpretation of both these formulas.

APPROXIMATION

It may be noticed that when $n \geq m$, $R(m, n) = 2^m$. On the other hand, if $m \gg n$ and for large n Cameron has developed an approximation for $N(m, n)$ which can be used to obtain a similar expression for $R(m, n)$. Under these conditions, Cameron finds that

$$N(m, n) \approxeq \sqrt{\frac{2}{\pi n}} \left(\frac{em}{n}\right)^n \tag{13}$$

and since

$$R(m, n) = \frac{1}{2} N(m+1, n+1) \tag{14}$$

under the same conditions

$$R(m, n) \cong \frac{1}{\sqrt{2\pi(n+1)}} \left(\frac{e(m+1)}{n+1}\right)^{n+1} \tag{15}$$

H. C. Joksch

D. M. Liss

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